Long time behaviour of a stochastic nano particle

Pierre Étoré*   Stéphane Labbé†   Jérôme Lelong‡

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Abstract

In this article, we are interested in the behaviour of a single ferromagnetic mono-domain particle submitted to an external field with a stochastic perturbation. This model is the first step toward the mathematical understanding of thermal effects on a ferromagnet. In a first part, we present the stochastic model and prove that the associated stochastic differential equation is well defined. The second part is dedicated to the study of the long time behaviour of the magnetic moment and in the third part we prove that the stochastic perturbation induces a non reversibility phenomenon. Last, we illustrate these results through numerical simulations of our stochastic model.

The main results presented in this article are on the one hand the rate of convergence of the magnetization toward the unique stable equilibrium of the deterministic model and on the other hand a sharp estimate of the hysteresis phenomenon induced by the stochastic perturbation (remember that with no perturbation, the magnetic moment remains constant).

Keywords: convergence rate, stochastic dynamical systems, long time study, magnetism, hysteresis.

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1 Introduction

Thermal effects in ferromagnetic materials are essential in order to understand their behaviour at ambient temperature or, more critically, in electronic devices where the joule effect induces high heat fluxes. This effect is commonly modeled by the introduction of a noise at micro-scopic scale on the magnetic moment direction and at the meso-scopic scale by a transition of behaviour. In ferromagnetic materials, the transition between the non-linear behaviour and the linear behaviour is managed by the struggle between the Heisenberg interaction and this disorder induced by the heating. This model explains the critical temperatures such as the Curie temperature for ferromagnetic materials. In this context, it is essential to understand the impact of introducing stochastic perturbations in deterministic models of ferromagnetic materials such as the micromagnetism (see Brown

*Grenoble INP, Laboratoire Jean Kuntzmann, 51 rue des Mathématiques, 38041 Grenoble cedex 9, France; pierre.etore@imag.fr

†Université Grenoble 1, Laboratoire Jean Kuntzmann, 51 rue des Mathématiques, 38041 Grenoble cedex 9, France; stephane.labbe@imag.fr (Grant HM-MAG, RTRA Foundation)

‡Grenoble INP, Laboratoire Jean Kuntzmann, 51 rue des Mathématiques, 38041 Grenoble cedex 9, France; jerome.lelong@imag.fr

(1962, 1963)).

The understanding of this phenomena is a key point in order to simulate realistic ferromagnetic devices such as micro electronic circuits. Furthermore, heating has a real effect on the microstructure dynamics in magnets; then, efficiently controlled, the dynamic of microstructures could accelerate processes such as the magnetization switching, which is the basics of magnetic recording techniques.

During the last decade, several studies have been initialized in several articles by physicists (e.g. Mercer et al. (2011); Zheng et al. (2003); Raikher et al. (2004); Atkinson et al. (2003); Raikher and Stepanov (2007); Scholz et al. (2001); Martinez et al. (2007); Smith (2001)), but no mathematical models justifying this kind of effects have been developed yet. In this article, our goal is to improve the understanding of thermal effects in ferromagnets. To achieve this goal, we focus this first study on the dynamic of a single magnetic moment submitted to a stochastic perturbation. Our main aim is to characterize precisely the dynamic of the moment, giving estimations and general behaviour in long time. In particular, we will exhibit an hysteresis behaviour of the magnetization in our model.

The model we are studying mimics the behaviour of a single magnetic moment $\mu(t)$ (function from $\mathbb{R}$ into $S(\mathbb{R}^3) = \{u \in \mathbb{R}^3;\ |\mu| = 1\}$) submitted to an external field b. The dynamic of such a system is, at the micro-scale, described by the Larmor precession equation

$$\frac{d\mu}{dt} = -\mu \wedge b.$$

Nevertheless, this equation is non dissipative and, in order to make the theoretical study easier, we introduce a dissipative part using the Landau-Lifchitz equation

$$\frac{d\mu}{dt} = -\mu \wedge b - \alpha\mu \wedge (\mu \wedge b),$$

where α is a positive real constant and we set the initial condition $\mu(0) = \mu_0 \in S(\mathbb{R}^3)$. We point out two major properties of this system

i. $\forall t \in \mathbb{R}, |\mu(t)| = 1$,

ii. $\forall t \in \mathbb{R},\ \dfrac{d}{dt}(\mu(t) \cdot b) \geq 0$.

The first property, which is definitely essential, will have to be preserved by the stochastic system and the second property is the energy decreasing induced by the introduction of the dissipation term. The dynamic of this deterministic system is classical; in fact, one knows that $\lim_{t\to\infty} \mu(t) = \dfrac{1}{|b|} b$ provided that $\mu(0) \neq -b$. In this work, we will develop such a result for the stochastic system. In order to build this stochastic system, the first question is how to introduce the stochastic perturbation in the deterministic system. We want to model the thermal effects which are external perturbations of the magnetic moment. In fact, this perturbation could be modeled has an external perturbation field. In the sequel, we will choose to build a stochastic system by perturbing the external field with a Brownian motion. We will write down

$$\begin{cases} dY_t & = -\mu_t \wedge (b\ dt + \varepsilon\ dW_t) - \alpha\mu_t \wedge \mu_t \wedge (b\ dt + \varepsilon\ dW_t) \\ Y_0 & = y \in \mathcal{S}(\mathbb{R}^3). \end{cases}$$

where ε is a strictly positive real number and W a standard Brownian motion with values in $\mathbb{R}^3$. But, an easy computation of $d(|Y_t|^2)$ using Itô's formula shows that the process Y will not stay in $S(\mathbb{R}^3)$, then, in order to preserve this essential behaviour, we have to renormalise the previous equation and set

$$\mu_t = \frac{Y_t}{|Y_t|}.$$

Given this system, we prove the following results

i. $\mu_t \cdot b \xrightarrow[t\to\infty]{} |b|$, a.s.,

ii. $\lim_{t\longrightarrow\infty} \sqrt{t}\,\mathbb{E}[||b| - \mu_t \cdot b|]$ exists and we can compute its value.

Note that these results are only valid for $\alpha > 0$, as for $\alpha = 0$,it is easy to show that the function $e(t) = \mathbb{E}(\mu_t \cdot b)$ satisfies the following ordinary differential equation $e'(t) = -e(t)\frac{h'(t)}{h(t)}$. Hence, $e(t) = \frac{e(0)}{h(t)} \xrightarrow[t\to\infty]{} 0$. This contradicts the a.s. convergence of μ_t to $\frac{b}{|b|}$.

iii. When μ is submitted to a time varying external field, an hysteresis phenomenon appears. If we consider $\boldsymbol{b} \in \mathcal{S}(\mathbb{R}^3)$ and let b linearly vary between $+\boldsymbol{b}$ and $-\boldsymbol{b}$ over the time interval $[0, T]$, then $\mathbb{E}(\mu_t \cdot \boldsymbol{b})$ is bounded from below by $\frac{1}{\sqrt{1+ct}}$ for $t \leq T/2$ where c is a constant depending only on ε and α.

First, we make precise the derivation of the stochastic model. Then, we lead a detailed study of its asymptotic behaviour and in particular we point out an hysteresis phenomenon. This phenomenon is obtained by slow variations of the external field such that the dynamic of relaxation of the magnetization toward this field becomes instantaneous when the speed ratio of the external excitation goes to zero. The results shown in this article are finally illustrated by numerical simulations.

2 Model and notations

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. We consider a standard Brownian motion W defined on this space with values in $\mathbb{R}^3$ and denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ its natural filtration augmented with $\mathbb{P}$null sets.

Let $b \in \mathbb{R}^3$ be the magnetic field. We model the $\mathcal{S}(\mathbb{R}^3)$-valued magnetic moment process $\mu = (\mu_t)_{t\geq 0}$ by the following coupled stochastic differential equation (SDE in short)

$$\begin{cases} dY_t &= -\mu_t \wedge (b\,dt + \varepsilon\,dW_t) - \alpha\mu_t \wedge \mu_t \wedge (b\,dt + \varepsilon\,dW_t) \\ \mu_t &= \frac{Y_t}{|Y_t|} \\ Y_0 &= y \in \mathcal{S}(\mathbb{R}^3), \end{cases} \tag{2.1}$$

where $\alpha > 0$ is the magnitude of the damping term and $\varepsilon > 0$ is the magnitude of the noise term.

The term $\mu_t \wedge dW_t$ in (2.1) is naturally defined by introducing the antisymmetric operator $L : \mathbb{R}^3 \longmapsto \mathbb{R}^{3\times 3}$ associated to the vector product in $\mathbb{R}^3$

$$L(x) = \begin{pmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & -x^1 \\ -x^2 & x^1 & 0 \end{pmatrix}.$$

Hence, for a 3-dimensional $\mathbb{F}-$adapted process H satisfying $\int_0^t |H_s|^2 \, ds < \infty$ a.s. for all $t > 0$, the process $\int_0^t H_s \wedge dW_s$ is defined by $\int_0^t L(H_s) dW_s$, which is a standard multi–dimensional Itô stochastic integral. When dealing with the differential expression, we will either write $H_t \wedge dW_t$ or $L(H_t)dW_t$.

Notations:

- For a and b in $\mathbb{R}^3$ we denote by $a \cdot b$ their scalar product, $a \cdot b = \sum_{i=1}^3 a^i b^i$.
- For a in $\mathbb{R}^3$, we denote by $|a| = \sqrt{a \cdot a}$ the Euclidean norm of a.
- We like to encode elements of $\mathbb{R}^3$ as column vectors. For $x \in \mathbb{R}^3$, x^* is a row vector. Similarly, we use the star notation "$*$" to denote the transpose of matrices.
- If $H = (H_t)_{t \geq 0}$ is a 3-dimensional $\mathbb{F}-$adapted process satisfying $\int_0^t |H_u|^2 du < \infty$ a.s. for all t, we may write $\int_0^t H_u \cdot dW_u$ for $\sum_{i=1}^3 \int_0^t H_u^i \, dW_u^i$ and use the differential form $H_t \cdot dW_t$ for $\sum_{i=1}^3 H_t^i \, dW_t^i$.

3 Main results: long time behaviour

3.1 First properties of the magnetic moment μ

Proposition 1. *Let (Y, μ) be a pair of processes satisfying* (2.1)*, then*

$$d\,|Y_t|^2 = 2\varepsilon^2(\alpha^2 + 1)dt$$

and therefore $|Y_t| = \sqrt{2\varepsilon^2(\alpha^2 + 1)t + 1}$ is non random.

Remark 2. *The fact that $|Y_t|$ is non random is definitely essential in all the following computations. In particular, we deduce from this result that $|Y_t|$ has finite variation.*

Proof. Using Itô's lemma we have

$$d\,|Y_t|^2 = 2Y_t \cdot dY_t + \sum_{i=1}^3 d\langle Y^i, Y^i \rangle_t = \sum_{i=1}^3 d\langle Y^i, Y^i \rangle_t,$$

where we have used the fact that Y_t and dY_t are orthogonal. But, using the identity $a \wedge (b \wedge c) = (a \cdot c)b - (a \cdot b)c$, we have

$$\begin{aligned} dY_t &= \varepsilon\big[-(\mu_t \wedge dW_t) - \alpha((\mu_t \cdot dW_t)\mu_t - (\mu_t \cdot \mu_t)dW_t)\big] + \ldots dt \\ &= \varepsilon A(\mu_t)\, dW_t + \ldots dt, \end{aligned}$$

where we have set $A(\mu) = \alpha I - \alpha(\mu\mu^*) - L(\mu)$ and used $|\mu_t| = 1$. Thus,

$$\begin{aligned} d\langle Y, Y \rangle_t &= \varepsilon^2 AA^*(\mu_t)dt \\ &= \varepsilon^2 \Big[\alpha^2 I - \alpha^2(\mu_t\mu_t^*) + \alpha L(\mu_t) - \alpha^2(\mu_t\mu_t^*) + \alpha^2(\mu_t\mu_t^*)(\mu_t\mu_t^*) - \alpha(\mu_t\mu_t^*)L(\mu_t) \\ &\qquad - \alpha L(\mu_t) + L(\mu_t)(\mu_t\mu_t^*) - L(\mu_t)L(\mu_t)\Big]\, dt \\ &= \varepsilon^2 \Big[\alpha^2 I - \alpha^2(\mu_t\mu_t^*) + L(\mu_t)L^*(\mu_t)\Big] dt, \end{aligned}$$

where we have used $L(\mu)\mu = 0$, $L^*(\mu) = -L(\mu)$ and again $\mu_t^*\mu_t = 1$. Thus, for each $1 \le i \le 3$ we have

$$d\langle Y^i, Y^i\rangle_t = \varepsilon^2\big[\alpha^2 - \alpha^2(\mu_t^i)^2 + \sum_{k=1}^{3}(L_{ik}(\mu_t))^2\big].$$

Then, summing over i, we get

$$\begin{aligned} d\,|Y_t|^2 &= \varepsilon^2\Big[3\alpha^2 - \alpha^2\,|\mu_t|^2 + \sum_{i,j=1}^{3}(L_{ij}(\mu_t))^2\Big]dt \\ &= \varepsilon^2[3\alpha^2 - \alpha^2\,|\mu_t|^2 + 2\,|\mu_t|^2]dt, \end{aligned}$$

The result ensues by remembering that $|\mu_t|^2 = 1$. ■

With the help of Proposition 1, we can establish the SDE satisfied by the one dimensional process $(\mu_t \cdot b)_t$. We introduce the function

$$h(t) = |Y(t)| = \sqrt{2\varepsilon^2(\alpha^2+1)t+1}. \tag{3.1}$$

Since $|Y_t|$ is non random, we deduce from Equation (2.1) that

$$\begin{aligned} d\mu_t &= -\frac{\mu_t h'(t)}{h(t)}dt + \frac{dY_t}{h(t)} \\ &= -\frac{\mu_t h'(t)}{h(t)}dt - \frac{1}{h(t)}\left(\mu_t \wedge (b\;dt + \varepsilon\;dW_t) + \alpha\mu_t \wedge \mu_t \wedge (b\;dt + \varepsilon\;dW_t)\right) \\ d\mu_t &= -\frac{\mu_t h'(t) + \mu_t \wedge b + \alpha(\mu_t(\mu_t\cdot b) - b)}{h(t)}dt - \frac{\varepsilon}{h(t)}\left(L(\mu_t) + \alpha(\mu_t\mu_t^* - I)\right)dW_t \end{aligned} \tag{3.2}$$

By taking the scalar product with b, we get

$$\begin{aligned} d(\mu_t\cdot b) &= -(\mu_t\cdot b)\frac{h'(t)}{h(t)}dt - \frac{\alpha}{h(t)}\left((\mu_t\cdot b)^2 - |b|^2\right)dt \\ &\quad - \frac{\varepsilon}{h(t)}\left((\mu_t \wedge\; dW_t)\cdot b + \alpha(\mu_t\cdot b)(\mu_t\cdot dW_t) - \alpha(b\cdot dW_t)\right) \\ d(\mu_t\cdot b) &= -(\mu_t\cdot b)\frac{h'(t)}{h(t)}dt - \frac{\alpha}{h(t)}\left((\mu_t\cdot b)^2 - |b|^2\right)dt \\ &\quad - \frac{\varepsilon}{h(t)}\big(-L(\mu_t)b + \alpha((\mu_t\cdot b)\mu_t - b)\big)\cdot dW_t \end{aligned} \tag{3.3}$$

We may call this equation the SDE satisfied by $\mu_t \cdot b$ whereas it is not an SDE properly speaking since the r.h.s member actually depends on all the components of μ_t and not only on $\mu_t \cdot b$. Nonetheless, we may use this abuse of terminology throughout the paper.

Remark 3 (Remark on the existence and uniqueness of solutions to Equation (2.1))**.** *Let us consider the following coupled SDE*

$$dY_t = -\mu_t \wedge (b\;dt + \varepsilon\;dW_t) - \alpha\mu_t \wedge \mu_t \wedge (b\;dt + \varepsilon\;dW_t) \tag{3.4a}$$

$$d\mu_t = -\frac{\mu_t h'(t) + \mu_t \wedge b + \alpha(\mu_t(\mu_t\cdot b) - b)}{h(t)}dt - \frac{\varepsilon}{h(t)}\left(L(\mu_t) + \alpha(\mu_t\mu_t^* - I)\right)dW_t \tag{3.4b}$$

$$Y_0 = \mu_0 \in \mathcal{S}(\mathbb{R}^3)$$

This system is actually decoupled as the SDE on μ is autonomous (this has only been possible because $|Y_t|$ is non random). The existence and uniqueness of a solution to Equation (3.4) *boil down to the ones of Equation* (3.4b)*. By computing* $d(|\mu_t|^2)$*, we deduce that if there exists a solution μ to Equation* (3.4b)*, $|\mu_t|^2 = 1$ a.s. for all t. Hence, it is sufficient to check the standard global Lipschitz behaviour of the coefficients on $\mathbb{R}_+ \times \mathcal{S}(\mathbb{R}^3)$ to prove existence and uniqueness of a strong solution to Equation* (3.4b)*.*
We have already seen above that if a pair (Y, μ) is solution of Equation (2.1)*, it also solves Equation* (3.4)*.*
Conversely, if (Y, μ) is the unique strong solution of Equation (3.4)*, it is clear that $|Y_t| = h(t)$ by following the proof of Proposition 1 and moreover the computation of $d(Y_t/|Y_t|)$ shows that the process $(Y_t/|Y_t|)_t$ solves the same SDE as μ, hence for all t $\mu_t = \frac{Y_t}{|Y_t|}$ a.s. This last argument proves that Equations* (2.1) *and* (3.4) *have the same solutions. Therefore, we deduce that Equation* (2.1) *admits a unique strong solution denoted by (Y, μ) in the sequel.*

3.2 Almost sure convergence

In this part, we prove the almost sure convergence of μ_t to $b/|b|$ when t goes to infinity. This is achieved by studying the pathwise behaviour of the process $(\mu_t \cdot b)_t$.

Theorem 4. $\lim\limits_{t \longrightarrow \infty} \mu_t \cdot b = |b|$ *a.s.*

To prove this result, we need a preliminary result stating that the stochastic integral in Equation (3.4b) actually vanishes at infinity.

Lemma 5.
$$\sup_t \int_0^t \frac{1}{h(u)} \big(-\mu_u \wedge b + \alpha((\mu_u \cdot b)\mu_u - b)\big) \cdot dW_u < \infty \quad a.s.$$

Proof of Theorem 4. From Lemma 5, we know that

$$\sup_t \int_0^t \frac{1}{h(u)} \big(-\mu_u \wedge b + \alpha((\mu_u \cdot b)\mu_u - b)\big) \cdot dW_u < \infty \quad a.s.$$

Hence, we can define for all $t \geq 0$

$$X_t = \mu_t \cdot b - \int_t^\infty \frac{\varepsilon}{h(u)} \big(-\mu_u \wedge b + \alpha((\mu_u \cdot b)\mu_u - b)\big) \cdot dW_u$$

Let $|b| > \delta > 0$ be chosen close to 0, there exists T such that for all $t \geq T$, $|X_t - \mu_t \cdot b| \leq \delta$. Moreover from Equation (3.3), we can deduce that for all $t > s > T$

$$X_t - X_s = \int_s^t -\mu_u \cdot b \frac{h'(u)}{h(u)} + \frac{\alpha}{h(u)}((\mu_u \cdot b)^2 - |b|^2) du. \tag{3.5}$$

Let $\eta < |b|$ be chosen close to $|b|$. On the set $\{0 < \mu_u \cdot b < \eta\}$ we have $|\mu_u \wedge b|^2 \geq |b|^2 - \eta^2$. Thus,

$$-(\mu_u \cdot b)\frac{h'(u)}{h(u)} + \frac{\alpha}{h(u)} |\mu_u \wedge b|^2 \geq -|b| \frac{h'(u)}{h(u)} + \alpha \frac{|b|^2 - \eta^2}{h(u)}.$$

We can always choose T such that for all $u > T$,

$$-|b|\frac{h'(u)}{h(u)} + \alpha\frac{|b|^2 - \eta^2}{h(u)} \geq \alpha\frac{|b|^2 - \eta^2}{2h(u)}.$$

On the set $\{\mu_u \cdot b \leq 0\}$ we have

$$\begin{aligned}-(\mu_u \cdot b)\frac{h'(u)}{h(u)} + \frac{\alpha}{h(u)}|\mu_u \wedge b|^2 &\geq (-(\mu_u \cdot b) + \alpha\,|\mu_u \wedge b|^2)\frac{h'(u)}{h(u)}\\ &\geq \min(|b|\,,\alpha\,|b|^2)\frac{1}{2}\frac{h'(u)}{h(u)}.\end{aligned}$$

The last inequality comes from the fact that if $\pi/2 \leq x \leq 3\pi/2$, we have either $-\cos(x) \geq \sqrt{2}/2$ or $|\sin(x)| \geq \sqrt{2}/2$.
Therefore, there exists $\bar{T} \geq T$, such that for all $u \geq \bar{U}$, on the event $\mu_u \cdot b < \eta$, $-(\mu_u \cdot b)\frac{h'(u)}{h(u)} + \frac{\alpha}{h(u)}|\mu_u \wedge b|^2$ is bounded from below by $c\frac{h'(u)}{h(u)}$ where c is a positive real constant depending on η. Therefore, we deduce from Equation (3.5)

$$\begin{aligned}X_t - X_s &\geq -|b|\int_s^t \frac{h'(u)}{h(u)}\,\mathbf{1}_{\{\mu_u \cdot > \eta\}}du + c\int_s^t \frac{h'(u)}{h(u)}\,\mathbf{1}_{\{\mu_u \cdot b \leq \eta\}}du\\ &\geq -|b|\int_s^t \frac{h'(u)}{h(u)}\,\mathbf{1}_{\{X_u > \eta-\delta\}}du + c\int_s^t \frac{h'(u)}{h(u)}\,\mathbf{1}_{\{X_u \leq \eta-\delta\}}du\end{aligned}$$

While choosing δ and η, we can always ensure that $0 < \eta - 2\delta < |b|$.
Assume that $X_s < \eta - 2\delta$. Then, there exists t for which the therm $\int_s^t \frac{h'(u)}{h(u)}\,\mathbf{1}_{\{X_u \leq \eta-\delta\}}du$ has driven X back into the cap such that $X_t > \eta - 2\delta$. Hence, we can assume that $X_s > \eta - 2\delta$. Either, for all t, $X_t > \eta - 2\delta$, or thanks to the continuity of X there exists t_0 for which $X_{t_0} = \eta - 2\delta$ and within a finite time the second integral drives X back into the polar cap such that at a some time $X_t \geq \eta - \delta$ and for all $u \in [t_0, t]$ $\eta - \delta \leq X_u > \eta - 2\delta$. This reasoning enables to prove to $X_t \geq \eta - 2\delta$, ie. $\mu_t \cdot b \geq \eta - 3\delta$. As η can chosen arbitrarily close to $|b|$ and δ arbitrarily small, this proves the almost convergence of $\mu_t \cdot b$ to $|b|$. ■

Proof of Lemma 5. From Doob's inequality we have

$$\mathbb{E}\left[\sup_t\left|\int_0^t \frac{1}{h(u)}\big(-\mu_u \wedge b + \alpha((\mu_u \cdot b)\mu_u - b)\big)\cdot dW_u\right|^2\right] \leq \int_0^\infty \frac{1}{h(u)^2}\mathbb{E}\left[(|b|^2 - |\mu_u \cdot b|^2)\right](1+\alpha^2)du. \tag{3.6}$$

Now, we will prove that the r.h.s is almost surely finite.
From Equation (3.3), we get after integrating and taking the expectation for all $t > 0$

$$\begin{aligned}\mathbb{E}[\mu_t \cdot b] - \mathbb{E}[\mu_0 \cdot b] &= -\int_0^t \mathbb{E}[\mu_u \cdot b]\frac{h'(u)}{h(u)} - \frac{\alpha}{h(u)}\mathbb{E}\left[(\mu_u \cdot b)^2 - |b|^2\right]du\\ \mathbb{E}[\mu_t \cdot b]'h(t) + \mathbb{E}[\mu_t \cdot b]h'(t) &= \frac{\alpha}{h(t)}\mathbb{E}\left[|b|^2 - (\mu_t \cdot b)^2\right]\\ \mathbb{E}[\mu_t \cdot b] - \frac{h(s)}{h(t)}\mathbb{E}[\mu_0 \cdot b] &= \frac{\alpha}{h(t)}\int_0^t \mathbb{E}\left[|b|^2 - (\mu_u \cdot b)^2\right]du\end{aligned}$$

Hence, we deduce that

$$\sup_t \frac{1}{h(t)} \int_0^t \mathbb{E}\left[|b|^2 - (\mu_u \cdot b)^2\right] du < \frac{2\,|b|}{\alpha} = \kappa.$$

Let us consider the upper–bound in Equation (3.6) truncated to t and perform an integration by parts to obtain

$$\begin{aligned}
&\int_0^t \frac{1}{h(u)^2}\mathbb{E}\left[(|b|^2 - |\mu_u \cdot b|^2)\right] du \\
&= \left[\frac{1}{h(u)^2}\int_0^u \mathbb{E}\left[(|b|^2 - |\mu_v \cdot b|^2)\right] dv\right]_0^t + \int_0^t \frac{2h'(u)}{h(u)^3}\int_0^u \mathbb{E}\left[(|b|^2 - |\mu_v \cdot b|^2)\right] dvdu \\
&\le \kappa \frac{1}{h(t)} + 2\kappa \int_0^t \frac{h'(u)}{h(u)^2} du \\
&\le \kappa\left(\frac{1}{h(t)} + 2\left(\frac{1}{h(0)} - \frac{1}{h(t)}\right)\right) \le \frac{2\kappa}{h(0)}
\end{aligned}$$

This proves that the r.h.s of Equation (3.6) is finite and ends the proof of Lemma 5. ∎

3.3 Convergence rate

In this section, we are in the behaviour of $h(t)(|b| - \mu_t \cdot b)$. In particular, we establish the rate of decrease of the $\mathbb{L}^1$ norm of $|b| - \mu_t \cdot b$ to zero.

Theorem 6. $\lim_{t\longrightarrow\infty} \mathbb{E}[h(t)\,||b| - \mu_t \cdot b|] = \dfrac{\varepsilon^2(1+\alpha^2)}{2\alpha}.$

Proof. As $|b| - \mu_t \cdot b \ge 0$, the $\mathbb{L}^1$ norm boils down to a basic expectation.
Let us define $\xi_t = |b| - \mu_t \cdot b$ for $t \ge 0$. From Equation (3.3), we get

$$\begin{aligned}
d\xi_t &= -\xi_t \frac{h'(t)}{h(t)} dt + |b| \frac{h'(t)}{h(t)} dt + \frac{\alpha}{h(t)}\left((\mu_t \cdot b)^2 - |b|^2\right) dt - \frac{\varepsilon}{h(t)}\big(-\mu_t \wedge b + \alpha((\mu_t \cdot b)\mu_t - b)\big) \cdot dW_t \\
&= -\xi_t \frac{h'(t)}{h(t)} dt + |b| \frac{h'(t)}{h(t)} dt - \frac{\alpha}{h(t)}\xi_t(2\,|b| - \xi_t) dt - \frac{\varepsilon}{h(t)}\big(-\mu_t \wedge b + \alpha((\mu_t \cdot b)\mu_t - b)\big) \cdot dW_t
\end{aligned}$$

If we introduce $Z_t = h(t)(|b| - \mu_t \cdot b)$, we can write

$$dZ_t = \left(h'(t)\,|b| - \alpha\xi_t(2\,|b| - \xi_t)\right) dt + \varepsilon\big(-\mu_t \wedge b + \alpha((\mu_t \cdot b)\mu_t - b)\big) \cdot dW_t$$

From the dynamics of Y, we deduce that $\mathbb{E}[Z_t]$ solves the following differential equation

$$\begin{aligned}
\mathbb{E}[Z_t]' &= |b|\,h'(t) - \alpha(2\,|b|\,\mathbb{E}[\xi_t] - \mathbb{E}[\xi_t^2]) \\
\mathbb{E}[Z_t]' &= |b|\,h'(t) - \alpha 2\,|b|\,\frac{\mathbb{E}[Z_t]}{h(t)} + \alpha\mathbb{E}[\xi_t^2] \\
\mathbb{E}[Z_t]' &= |b|\,h'(t) - \alpha\frac{2\,|b|}{\varepsilon^2(\alpha^2+1)} h'(t)\mathbb{E}[Z_t] + \alpha\mathbb{E}[\xi_t^2] \\
\left(\mathbb{E}[Z_t]\, e^{\int_0^t \frac{2\alpha|b|}{\varepsilon^2(\alpha^2+1)} h'(u)du}\right)' &= \left(|b|\,h'(t) + \alpha\mathbb{E}[\xi_t^2]\right) e^{\int_0^t \frac{2\alpha|b|}{\varepsilon^2(\alpha^2+1)} h'(u)du}
\end{aligned}$$

Now, we can integrate the previous equation to obtain

$$\begin{aligned}
\mathbb{E}[Z_t]\,\mathrm{e}^{\frac{2\alpha|b|}{\varepsilon^2(\alpha^2+1)}(h(t)-h(0))} - \mathbb{E}[Z_0] &= |b| \int_0^t h'(u)\,\mathrm{e}^{\frac{2\alpha|b|}{\varepsilon^2(\alpha^2+1)}(h(u)-h(0))}\,du \\
&\quad + \alpha \int_0^t \mathbb{E}[\xi_u^2]\,\mathrm{e}^{\frac{2\alpha|b|}{\varepsilon^2(\alpha^2+1)}(h(u)-h(0))}\,du \\
\mathbb{E}[Z_t] - \mathbb{E}[Z_0]\,\mathrm{e}^{-\frac{2\alpha|b|}{\varepsilon^2(\alpha^2+1)}(h(t)-h(0))} &= \frac{\varepsilon^2(\alpha^2+1)}{2\alpha}\left(1 - \mathrm{e}^{-\frac{2\alpha|b|}{\varepsilon^2(\alpha^2+1)}(h(t)-h(0))}\right) \\
&\quad + \alpha\,\mathrm{e}^{-\frac{2\alpha|b|}{\varepsilon^2(\alpha^2+1)}(h(t)-h(0))} \int_0^t \mathbb{E}[\xi_u^2]\,\mathrm{e}^{\frac{2\alpha|b|}{\varepsilon^2(\alpha^2+1)}(h(u)-h(0))}\,du
\end{aligned}$$

From Theorem 4, we know that ξ_t tends to 0 a.s., therefore the bounded convergence theorem yields that $\lim_{u\longrightarrow\infty} \mathbb{E}[\xi_u^2] = 0$. Hence, as $h(t)$ tends to infinity with t, it is easy to show that

$$\lim_{t\longrightarrow\infty} \mathrm{e}^{-\frac{2|b|}{\varepsilon^2(\alpha^2+1)}(h(t)-h(0))} \int_0^t \mathbb{E}[\xi_u^2]\,\mathrm{e}^{\frac{2|b|}{\varepsilon^2(\alpha^2+1)}(h(u)-h(0))}\,du = 0.$$

Then, we can deduce that

$$\lim_{t\longrightarrow\infty} \mathbb{E}[Z_t] = \frac{\varepsilon^2(\alpha^2+1)}{2\alpha}.$$

■

As a corollary of Theorem 6, we can prove the following results using Markov's inequality.

Corollary 7. *For all* $0 < \beta < 1/2$ *and* $\eta > 0$, $\mathbb{P}(t^\beta(|b| - \mu_t \cdot b) \geq \eta) \longrightarrow 0$.

4 Hysteresis phenomena

In this section, we want to study the impact of the stochastic perturbation on the reversibility of the system; we are wondering whether the stochastic part may induce an hysteresis phenomenon. In order to observe this, the particle is submitted to an external field linearly varying from $+\boldsymbol{b}$ to $-\boldsymbol{b}$ where $\boldsymbol{b} \in \mathcal{S}(\mathbb{R}^3)$ and with constant direction and bounded modulus. We have seen in Section 3 that when the external field is fixed, the magnetic moment μ asymptotically stabilizes along this field. If the external field varies sufficiently slowly compared to the stabilization rate of μ, we expect that μ will take different back an forth paths when the external field switches from $+\boldsymbol{b}$ to $-\boldsymbol{b}$ and then from $-\boldsymbol{b}$ to $+\boldsymbol{b}$: this is the non reversibility property of the system.
In order to highlight this property, we will study the evolution of a suitably rescaled system on the time interval $[0, 1]$ and show that the average back and forth paths of $\mu_t \cdot b$ can not cross at the point $t = 1/2$.

We consider a two time scale model: a slower scale for the variations of the external field and a faster scale for the Landau Lifshitz evolution of the magnetic moment.
Let $\eta > 0$ be a fixed time scale and $\boldsymbol{b} \in \mathcal{S}(\mathbb{R}^3)$ the direction of the external field. We define the external filed b^η linearly varying between $+\boldsymbol{b}$ and $-\boldsymbol{b}$ on the interval $[0, 1/\eta]$ by

$$b^\eta(t) = (1 - 2t\,\eta)\,\boldsymbol{b} \quad \text{for} t \in [0, 1/\eta]$$

We assume that the magnetic moment μ^η is affected by $b^\eta(t)$ according to the following equation for $t \in [0, 1/\eta]$

$$\begin{cases} dY_t^\eta &= -\mu_t^\eta \wedge (b^\eta(t)\, dt + \varepsilon\, dW_t) - \alpha \mu_t^\eta \wedge \mu_t^\eta \wedge (b^\eta(t)\, dt + \varepsilon\, dW_t) \\ \mu_t^\eta &= \frac{Y_t^\eta}{|Y_t^\eta|} \\ Y_0^\eta &= \boldsymbol{b} \end{cases}$$

In order to work on the interval $[0, 1]$, we introduce rescaled versions of both the external field and the magnetic moment defined for $t \in [0, 1]$.

$$b(t) = b^\eta(t/\eta), \qquad Z_t = Y_{t/\eta}^\eta, \qquad \lambda_t = \mu_{t/\eta}^\eta.$$

Using the time scale property of the stochastic integral, we can write

$$dZ_t = -\lambda_t \wedge \left(b(t)\, \frac{1}{\eta} dt + \varepsilon\, dW_{t/\eta} \right) - \alpha \lambda_t \wedge \lambda_t \wedge \left(b(t)\, \frac{1}{\eta} dt + \varepsilon\, dW_{t/\eta} \right)$$

From the scaling property of the Brownian motion, we know that $(\sqrt{\eta} W_{t/\eta})$ is still a Brownian motion. So we get

$$dZ_t = -\lambda_t \wedge \left(b(t)\, \frac{1}{\eta} dt + \varepsilon\, \frac{1}{\sqrt{\eta}} dW_t \right) - \alpha \lambda_t \wedge \lambda_t \wedge \left(b(t)\, \frac{1}{\eta} dt + \varepsilon \frac{1}{\sqrt{\eta}}\, dW_t \right)$$

It is important to notice that the factor η acts as a time scale parameter for the deterministic part, but that the corresponding scaling parameter for the stochastic part is $\sqrt{\eta}$.
Following the proof of Proposition 1, it is obvious to show that $d\,|Z_t|^2 = 2(1+\alpha^2)\varepsilon^2/\eta\ dt$. Then, we introduce for $t \in [0, 1]$

$$h^\eta(t) = \sqrt{2(1+\alpha^2)\varepsilon^2 t/\eta + 1}.$$

The Ito formula applied to $(\lambda_t \cdot \boldsymbol{b})_t$ yields

$$\begin{aligned} d(\lambda_t \cdot \boldsymbol{b}) = &-(\lambda_t \cdot \boldsymbol{b}) \frac{h^{\eta\prime}(t)}{h^\eta(t)} dt + \alpha \frac{1-2t}{\eta h^\eta(t)}\, |\lambda_t \wedge \boldsymbol{b}|^2\, dt \\ &- \frac{\varepsilon}{\sqrt{\eta} h^\eta(t)} \left((\lambda_t \wedge\ dW_t) \cdot \boldsymbol{b} + \alpha (\lambda_t \cdot \boldsymbol{b})(\lambda_t \cdot dW_t) - \alpha (\boldsymbol{b} \cdot dW_t) \right) \end{aligned}$$

The stochastic part vanishes through when taking expectation as in the previous section to find

$$\mathbb{E}(\lambda_t \cdot \boldsymbol{b}) - \lambda_0 \cdot \boldsymbol{b} = \int_0^t -\mathbb{E}(\lambda_u \cdot \boldsymbol{b}) \frac{h^{\eta\prime}(u)}{h^\eta(u)} + \alpha \frac{1-2u}{\eta h^\eta(u)} \mathbb{E}\, |\lambda_u \wedge \boldsymbol{b}|^2\, du \tag{4.1}$$

Proposition 8. *For all $t \in [0, 1/2]$,*

$$\mathbb{E}(\lambda_t \cdot \boldsymbol{b}) \geq \frac{1}{h^\eta(t)} \geq \frac{1}{\sqrt{1 + \frac{(1+\alpha^2)\varepsilon^2}{\eta}}}.$$

Proof. Since the process λ is pathwise continuous and bounded, it is easy to show that the deterministic function $e(t) : t \longmapsto \mathbb{E}(\lambda_t \cdot b)$ is of class C^1. Hence, we can differentiate Equation (4.1)

$$e'(t) = -e(t)\frac{h^{\eta\prime}(t)}{h^\eta(t)} + \alpha\frac{1-2t}{\eta h^\eta(t)}\mathbb{E}\,|\lambda_t \wedge \boldsymbol{b}|^2$$

As $t \leq 1/2$, the second term on the r.h.s is non–negative, hence

$$e'(t) \geq -e(t)\frac{h^{\eta\prime}(t)}{h^\eta(t)}$$

From this inequality, we deduce that $(e(t)h^\eta(t))' \geq 0$, which leads to the following lower bound

$$e(t) \geq \frac{1}{h^\eta(t)} \quad \text{for } t \leq 1/2$$

■

5 Numerical experiments

In this section, we want to illustrate the theoretical results obtained in Sections 3 and 4 using some numerical simulations.

Long time behaviour. We consider the stochastic system (2.1) and discretize it with the help of an Euler scheme $(\bar{Y}, \bar{\mu})$, on a time grid with step size $\delta t > 0$.
Figure 1 shows the long time behaviour of $(\bar{\mu}_t \cdot b)_{t\geq 0}$, for one path of the scheme $(\bar{Y}, \bar{\mu})$, with time step size $\delta t = 0.01$ and for different values of the damping parameter α. The parameter ε is fixed to 0.1, we have taken $|b| = 1$, and set $\mu_0 = -b$. The almost sure convergence of $\mu_t \cdot b$ to $|b|$, as stated by Theorem 4, is well illustrated by Figure 1 and one can also see how the parameter α impacts the characteristic time of the system, ie. the time needed to stabilize around the limit.
Now, we wish to compare the rates of convergence studied in Subsection 3.3 to numerical observations. From Theorem 6, $\frac{2\alpha\sqrt{2}}{\varepsilon\sqrt{(1+\alpha^2)}}\sqrt{t}\ \mathbb{E}(|b| - \mu_t.b)$ converges to 1 when t goes to infinity. This is illustrated by Figure 2 for different values of α. This figure confirms that decreasing the parameter α leads to a decrease of the convergence rate of $\mathbb{E}(|b| - \mu_t.b)$.

Hysteresis phenomena. On Figure 3, we can observe a typical pathwise hysteresis phenomenon, which not only illustrates Proposition 8 but also suggests that the result of this Proposition could well improved by proving a almost sure lower bound (probably for sufficiently small values η). On Figure 3, the forward path (red curve) is almost stuck to the value 1 on the interval $[0, 1]$, we could then be tempted to think that the lower bound of Proposition 8 lacks some accuracy.
On the contrary, when η becomes small, which corresponds to a slower scale for the variations of the external field, Figures 4 and 5 show that the lower bound $1/h^\eta(t)$ becomes nearly optimal for t lower but close to $1/2$.

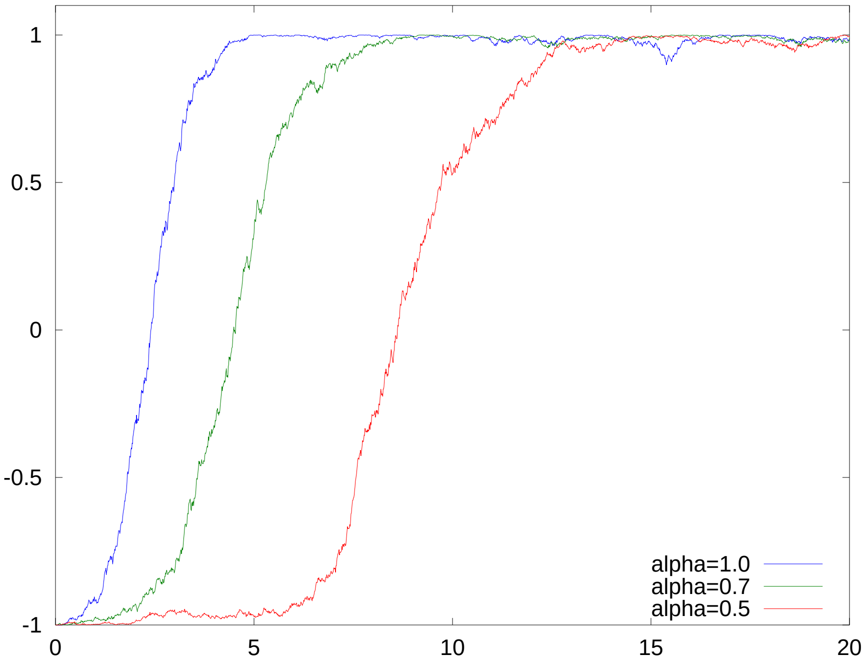


Figure 1: Almost sure convergence of $\mu_t \cdot b$ with $\mu_0 = -b$, $|b| = 1$, $\varepsilon = 0.1$.

6 Conclusion

In this article, we analyzed the long time behaviour of a non linear SDE modeling the evolution of a magnet submitted to a perturbated external field. The rate of convergence of the magnetic moment is particularly interesting and holds for any dissipation coefficient $\alpha > 0$. This result has been obtained by combining the ODE technique with Itô's formula. The second result concerns the hysteresis behaviour of the system induced by the stochastic perturbation. The combination of these two results illustrated the dissipative effects of the stochastic perturbation on the system by giving a first glimpse on thermal effects on a ferromagnet can be modeled in the framework of micromagnetism.

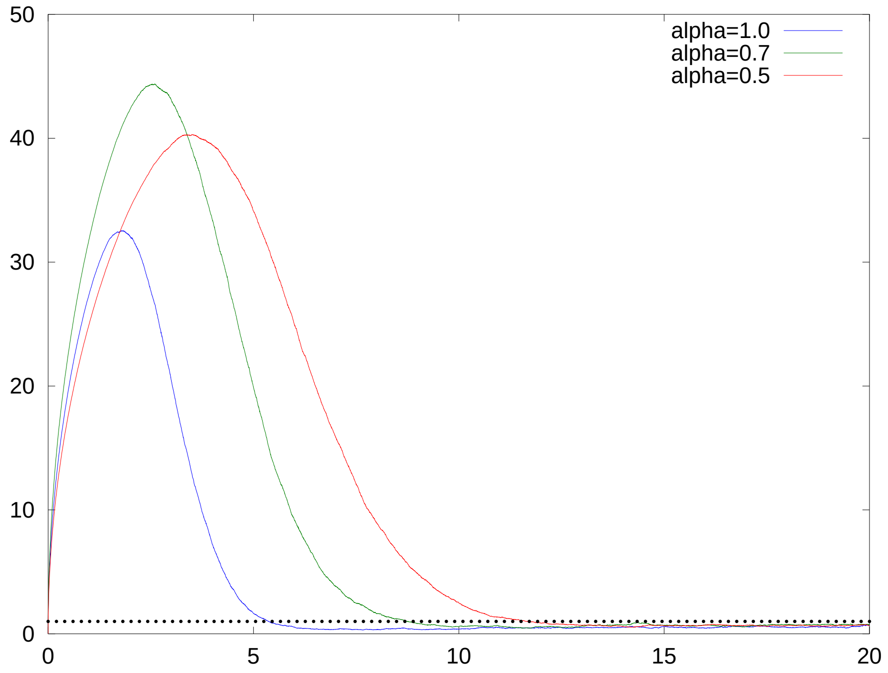


Figure 2: Convergence of $\frac{2\alpha\sqrt{2}}{\varepsilon\sqrt{(1+\alpha^2)}}\sqrt{t}\ \mathbb{E}(|b| - \mu_t.b)$ with $\mu_0 = -b$, $|b| = 1$ and $\varepsilon = 0.1$. The horizontal dashed line is at level one. The expectation is computed using a Monte–Carlo method with 100 samples.

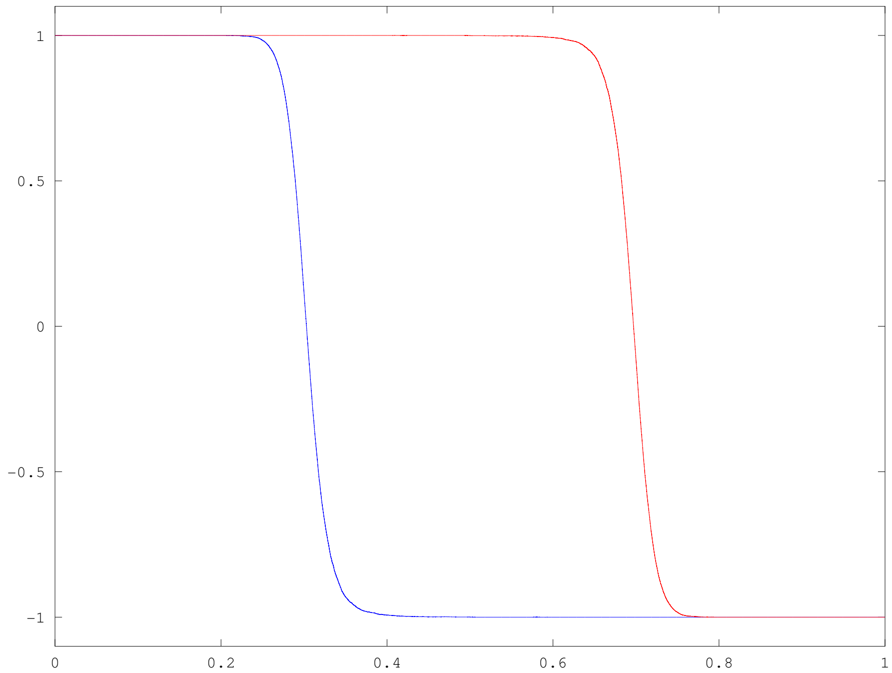


Figure 3: Pathwise hysteresis phenomena with $\alpha = 1$, $\varepsilon = 0.005$ and $\eta = 0.01$. The red curve is the forward path whereas the blue curve is the backward path.

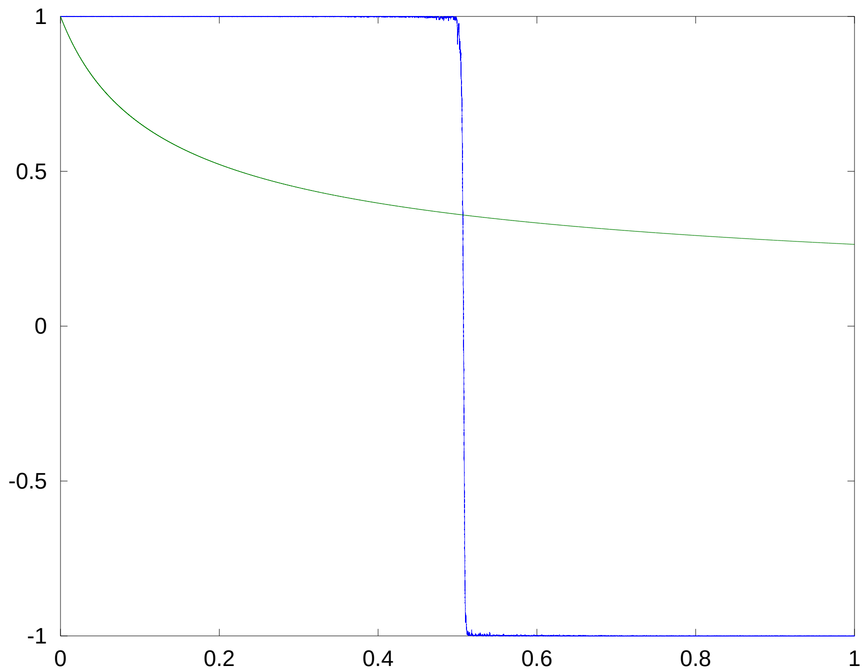


Figure 4: Pathwise hysteresis phenomena with $\alpha = 1$, $\varepsilon = 0.01$ and $\eta = 3.1E - 5$. The blue curve is the evolution of $\mu_t \cdot \boldsymbol{b}$ and the green curve is $1/h^\eta(t)$.

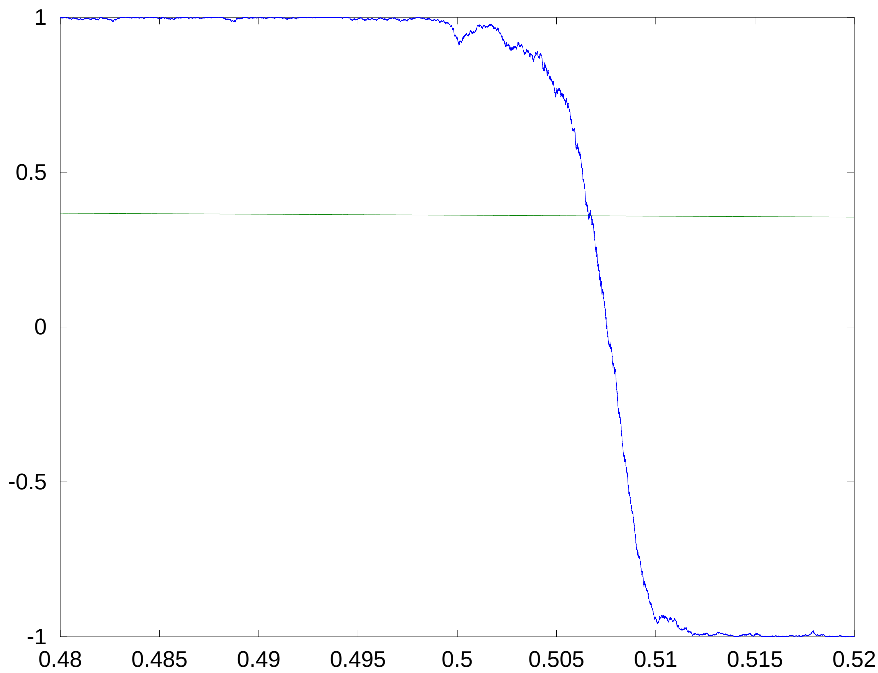


Figure 5: Zoom of Figure 4 around $t = 1/2$. The blue curve is the evolution of $\mu_t \cdot \boldsymbol{b}$ and the green curve is $1/h^\eta(t)$.

References

D. Atkinson, D. A. Allwood, C. C. Faulkner, G. Xiong, M. D. Cooke, and R. P. Cowburn. Magnetic domain wall dynamics in a permalloy nanowire. *IEEE Transactions on Magnetics*, 39(5):2663–2665, September 2003.

W. Brown. *Magnetostatic Principles in Ferromagnetism*. North-Holland, 1962.

W.-F. Brown. *Micromagnetics*. Interscience Publishers, 1963.

E. Martinez, L. Lopez-Diaz, L. Torres, and C. Garcia-Cervera. Micromagnetic simulations with thermal noise: Physical and numerical aspects. *Journal of Magnetism and Magnetic Materials*, 316:269–272, 2007.

J. I. Mercer, M. L. Plumer, J. P. Whitehead, and J. van Ek. Atomic level micromagnetic model of recording media switching at elevated temperatures. *Applied Physics Letters*, 98, 2011.

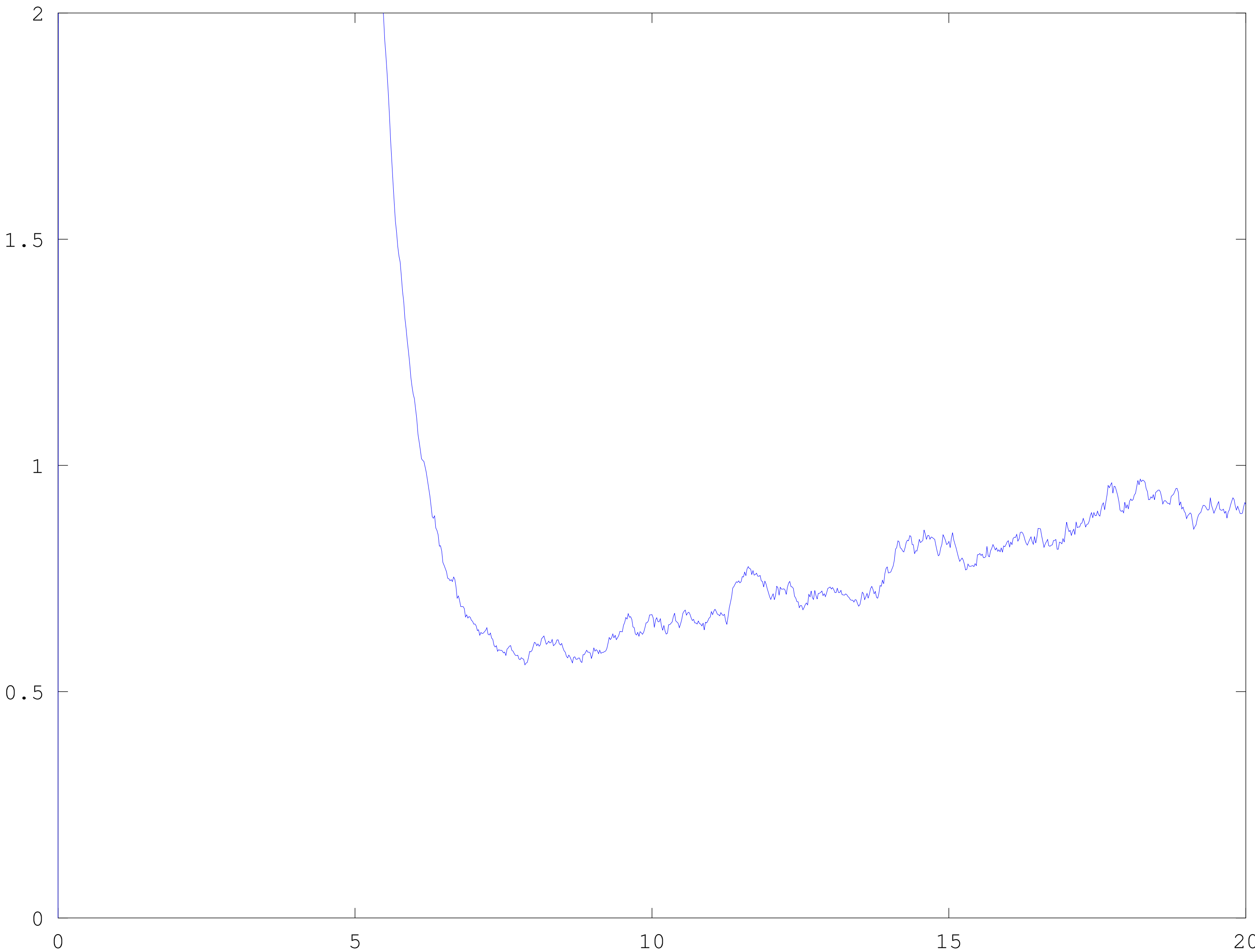
Y. Raikher and V. Stepanov. Magnetization dynamics of single-domain particles by superparamagnetic theory. *Journal of Magnetism and Magnetic Materials*, 316:417–421, 2007.

Y. Raikher, V. Stepanov, and R. Perzynskib. Dynamic hysteresis of a superparamagnetic nanoparticle. *Physica B*, 343:262–266, 2004.

W. Scholz, T. Schrefl, and J. Fidler. Micromagnetic simulation of thermally activated switching in fine particles. *Journal of Magnetism and Magnetic Materials*, 233:296–304, 2001.

N. Smith. Modeling of thermal magnetization fluctuations in thin-film magnetic devices. *Journal of Applied Physics*, 90:5768–5773, 2001.

G.-P. Zheng, D. Gross, and M. Li. Atomistic modeling of nanocrystalline ferromagnets. *Journal of Applied Physics*, 93(10):7652–7654, May 2003.



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